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Technical Report No.11

THE PROPAGATION OF ERRORS, FLUCTUATIONS AND TOLERANCES

- SUPPLEMENTARY FORMULAS

by

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* Essentially equivalent material will also appear in an internal Technical Memorandum at Bell Telephone Laboratories, Inc.

Contents

This technical report supplements Technical Report No.10, "The Propagation of Errors, Fluctuations and Tolerances - Basic Generalized Formulas"

45. Cumulants when independent variables need not be statistically independent

It has been suggested (by D. H. Evans) that the application of the generalized propagation formulas to components characterized by several quantities such as tubes and transistors will require dropping the assumption of statistical independence for the corresponding variables. It was remarked (at page II-10 of the first of this series) that only terms of order σ^3, or perhaps σ^4, are likely to be written down. We shall shortly give the formulas through σ^3, reserving the σ^4 terms for a later section.

Before we can meaningfully write down the formulas, we must introduce more notation for moments - there may now be many more nonvanishing moments! To the standard

$$\text{ave } (w_a - \bar{w}_a)^2 = \sigma_a^2$$

and

$$\text{ave } (w_a - \bar{w}_a)(w_b - \bar{w}_b) = \rho_{ab}\sigma_a\sigma_b, \quad (a \neq b)$$

we add

$$\text{ave}\ (w_a-\bar{w}_a)(w_b-\bar{w}_b)(w_c-\bar{w}_c) = \gamma_{abc}\sigma_a\sigma_b\sigma_c$$

and

$$\text{ave}\ (w_a-\bar{w}_a)(w_b-\bar{w}_b)(w_c-\bar{w}_c)(w_d-\bar{w}_d) = \Gamma_{abcd}\sigma_a\sigma_b\sigma_c\sigma_d.$$

(In the latter two formulas a, b, c and d can be alike or different in any combination.)

It is important to compare the number of distinct moments which are to be taken into account in the independent and non-independent cases. This is done in Table 9, which shows clearly why the non-independent case becomes rather unmanageable so easily. (In the general case, for k = 20, say, we have, in the independent and general cases, respectively, 40 or 1750 moment constants through third moments, and 60 or 10605 through fourth moments.) As long as dependence only exists within groups of a small number of parameters, the number of moment-constants needed will not grow nearly as fast as the values in Table 2, and the correction terms will be usable for larger k.

The formulas through terms in σ^3 are given in Table 10. (For terms in σ^4, see section 47.)

Table 9

Numbers of moments of each order and kind which must be considered for k variables.

Order	Kind	Variables Independent	Variables Nonindependent
Second	σ_a^2	k	k
	$\rho_{ab}\sigma_a\sigma_b$	-	$\frac{1}{2}k(k-1)$
(total to this point)		(k)	$(\frac{1}{2}k(k+1))$
Third	$\gamma_{aaa}\sigma_a^3$	k	k
	$\gamma_{aab}\sigma_a^2\sigma_b$	-	$k(k-1)$
	$\gamma_{abc}\sigma_a\sigma_b\sigma_c$	-	$\frac{1}{6}k(k-1)(k-2)$
(total to this point)		(2k)	$(\frac{1}{6}k(k+1)(k+5))$
Fourth	$\Gamma_{aaaa}\sigma_a^4$	k	k
	$\Gamma_{aaab}\sigma_a^3\sigma_b$	-	$k(k-1)$
	$\Gamma_{aabb}\sigma_a^2\sigma_b^2$	-	$\frac{1}{2}k(k-1)$
	$\Gamma_{aabc}\sigma_a^2\sigma_b\sigma_c$	-	$\frac{1}{2}k(k-1)(k-2)$
	$\Gamma_{abcd}\sigma_a\sigma_b\sigma_c\sigma_d$	-	$\frac{1}{24}k(k-1)(k-2)(k-3)$
(total to this point)		(3k)	$(\frac{1}{24}k(k+1)(k^2+9k+26)$

Table 10

Propagation formulas through σ^3 for $z = h(w_1, w_2, \ldots, w_k)$ where the w_a are not necessarily independent.

$$\text{ave } z = h(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k)$$
$$+ \frac{1}{2}\Sigma h_{aa}\sigma_a^{\ 2} + \Sigma^* h_{ab}\rho_{ab}\sigma_a\sigma_b$$
$$+ \frac{1}{6}\Sigma h_{aaa}\gamma_{aaa}\sigma_a^{\ 3} + \frac{1}{2}\Sigma^* h_{aab}\gamma_{aab}\sigma_a^{\ 2}\sigma_b$$
$$+ \Sigma^* h_{abc}\gamma_{abc}\sigma_a\sigma_b\sigma_c$$
$$+ \text{terms of order} \geq \sigma^4$$

$$\text{var } z = \Sigma h^2\sigma_a^{\ 2} + 2\Sigma^* h_a h_b\rho_{ab}\sigma_a\sigma_b \quad \text{(classical)}$$
$$+ \Sigma h_a h_{aa}\gamma_{aaa}\sigma_a^{\ 3} + \Sigma h_a h_{bb}\gamma_{abb}\sigma_a\sigma_b^{\ 2}$$
$$+ 2\Sigma^* h_a h_{ab}\gamma_{aab}\sigma_a^{\ 2}\sigma_b + 2\Sigma^* h_a h_{bc}\gamma_{abc}\sigma_a\sigma_b\sigma_c$$
$$+ \text{terms of order} \geq \sigma^4$$

$$\text{ske } z = \Sigma h_a^{\ 3}\gamma_{aaa}\sigma_a^{\ 3} + 3\Sigma^* h_a^{\ 2}h_b\gamma_{aab}\sigma_a^{\ 2}\sigma_b$$
$$+ 6\Sigma^* h_a h_b h_c\gamma_{abc}\sigma_a\sigma_b\sigma_c$$
$$+ \text{terms of order} \geq \sigma^4$$

$$\text{elo } z = \text{terms of order} \geq \sigma^4$$

(Σ^* signifies summation over distinct, distinguishable terms, see the first of this series for details).

46. Cocumulents for statistical independence

There may be cases where we wish to deal simultaneously with

$$z = h(w_1, w_2, \ldots, w_k)$$

and

$$y = g(w_1, w_2, \ldots, w_k)$$

where the w's are statistically independent. To the formulas for propagation into the cumulants of z and y alone, we may wish to add formulas for propagation into their cocumulants. The resulting formulas are given in Table 11 through terms in σ^4. (Note the use of γ_a and Γ_a rather than γ_{aaa} and Γ_{aaaa} to emphasize that we are assuming statistical independence.) The manner of formation of these terms from those of the corresponding cumulant formulas is quite simple. For example,

$$6h_a h_b h_{ab}$$

would naturally be replaced by

$$f_a g_b h_{ab} + f_a h_b g_{ab} + g_a f_b h_{ab} + g_a h_b f_{ab}$$

$$+ h_a f_b g_{ab} + h_a g_b f_{ab},$$

but when appearing as the coefficient of something symmetric in a and b, the latter can be replaced by

$$f_a g_b h_{ab} + f_a h_b g_{ab} + g_a h_b f_{ab}$$

Table 11

Formulas for propagation into cocumulants of $z = h(w_1, w_2, \ldots, w_k)$ and $y = g(w_1, w_2, \ldots, w_k)$ and $x = f(w_1, w_2, \ldots, w_k)$ from statistically independent w's.

$$
\begin{aligned}
\operatorname{cov}(y,z) = {} & \Sigma g_a h_a \sigma_a^2 \\
& + \tfrac{1}{2}\Sigma(h_a g_{aa} + g_a h_{aa})\gamma_a \sigma_a^3 \\
& + \tfrac{1}{6}\Sigma(h_a g_{aaa} + g_a h_{aaa})\Gamma_a \sigma_a^4 + \tfrac{1}{4}\Sigma g_{aa} h_{aa}(\Gamma_a - 1)\sigma_a^4 \\
& + \Sigma^*(\tfrac{1}{2} g_a h_{abb} + \tfrac{1}{2} h_a g_{abb} + h_{ab} g_{ab} \\
& \qquad + \tfrac{1}{2} h_{aab} g_b + \tfrac{1}{2} g_{aab} h_b)\sigma_a^2 \sigma_b^2 \\
& + \text{terms of order} \geq \sigma^5
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cok}(x,y,z) = {} & \Sigma f_a g_a h_a \sigma_a^3 \\
& + \tfrac{1}{2}\Sigma(f_a g_a h_{aa} + f_a h_a g_{aa} + g_a h_a f_{aa})(\Gamma_a - 1)\sigma_a^4 \\
& + 2\Sigma^*(f_a g_b h_{ab} + f_a h_b g_{ab} + g_a h_b f_{ab})\sigma_a^2 \sigma_b^2 \\
& + \text{terms of order} \geq \sigma^5
\end{aligned}
$$

(symmetry of h_{ab}, g_{ab} etc. in their subscripts to be recognized in interpreting Σ^*)

in view of the identity (for unsymmetric A_{ab})

$$\Sigma^*(A_{ab} + A_{ba})\, u_a u_b \equiv \Sigma^* A_{ab} u_a u_b$$

The basic process is to replace the various h-factors in the propagation-into-cumulant formula by different g- and h-factors (or f-, g- and h-factors) in all possible ways, sum and adjust the coefficient. The correctness of the result is easily checked, since

(i) every term must have a factor of each appropriate kind

(ii) there must be symmetry under the interchange of arguments in the cocumulant

(iii) when all arguments are made the same, the result must reduce to the corresponding cumulant result.

Extensions to Table 11 can thus be obtained easily when desired.

47. Reconversion formulas

Now that propagation formulas from nonindependent variables and into cocumulants are available, it is not unlikely that one may be used to supply input to the other. It is thus convenient to record here the formulas expressing reduced (= standardized) moments in terms of cumulants and cocumulants, namely:

$$\rho_{ab} = \frac{\text{cov}\ (w_a, w_b)}{\sigma_a \sigma_b}$$

$$\gamma_{aaa} = \frac{\text{ske}\ w_a}{\sigma_a^3}$$

$$\gamma_{abc} = \frac{\text{cok}\ (w_a, w_b, w_c)}{\sigma_a \sigma_b \sigma_c}$$

$$\Gamma_{aaaa} = \frac{\text{elo}\ w_a}{\sigma_a^4} + 3$$

$$\Gamma_{abcd} = \frac{\text{coe}\ (w_a, w_b, w_c, w_d)}{\sigma_a \sigma_b \sigma_c \sigma_d} + \rho_{ab}\rho_{cd} + \rho_{ac}\rho_{bd} + \rho_{ad}\rho_{bc}\ .$$

(Any identification desired among a,b,c or d may be made freely.)

48. Detailed formulas for the statistically non-independent case

We now record for possible use the formulas through terms of order σ^4 for ave z, var z, ske z and elo z. These are based on the formulas for the covariances and coskewnesses of (not necessarily independent) centered monomials which are given in Tables 12 and 13. The derivation follows exactly the lines of Part V.

The formulas for the propagation into cumulants follow.

$$
\begin{aligned}
\text{ave } z \ = \ & h\,(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k) \\
& + \tfrac{1}{2}\,\Sigma h_{aa}\sigma_a{}^2 + \Sigma^* h_{ab}\rho_{ab}\sigma_a\sigma_b \\
& + \tfrac{1}{6}\,\Sigma h_{aaa}\gamma_{aaa}\sigma_a{}^3 + \tfrac{1}{2}\,\Sigma^* h_{aab}\gamma_{aab}\sigma_a{}^2\sigma_b \\
& \qquad + \Sigma^* h_{abc}\gamma_{abc}\sigma_a\sigma_b\sigma_c \\
& + \tfrac{1}{24}\,\Sigma h_{aaaa}\Gamma_{aaaa}\sigma_a{}^4 + \tfrac{1}{6}\,\Sigma^* h_{aaab}\Gamma_{aaab}\sigma_a{}^3\sigma_b \\
& \qquad + \tfrac{1}{4}\,\Sigma^* h_{aabb}\Gamma_{aabb}\sigma_a{}^2\sigma_b{}^2 + \tfrac{1}{2}\,\Sigma^* h_{aabc}\Gamma_{aabc}\sigma_a{}^2\sigma_b\sigma_c \\
& \qquad + \Sigma^* h_{abcd}\Gamma_{abcd}\sigma_a\sigma_b\sigma_c\sigma_d \\
& + \text{terms of order} \geq \sigma^5
\end{aligned}
$$

Table 12

Table of coefficients by which $\sigma_a^{i+m}\,\sigma_b^{j+n}$ must be multiplied to obtain the covariance of $w_a^{i}\,w_b^{j}$ with $w_a^{m}\,w_b^{n}$ when w_a and w_b need not be statistically independent, but both have average zero.

	w_a	w_b	w_a^2	$w_a w_b$	w_b^2
w_a	1	ρ_{ab}	γ_{aaa}	γ_{aab}	γ_{abb}
w_b	ρ_{ab}	1	γ_{aab}	γ_{abb}	γ_{bbb}
w_a^2	γ_{aaa}	γ_{aab}	$\Gamma_{aaaa}-1$	$\Gamma_{aaab}-\rho_{ab}$	$\Gamma_{aabb}-1$
$w_a w_b$	γ_{aab}	γ_{abb}	$\Gamma_{aaab}-\rho_{ab}$	$\Gamma_{aabb}-\rho_{ab}^2$	$\Gamma_{abbb}-\rho_{ab}$
w_b^2	γ_{abb}	γ_{bbb}	$\Gamma_{aabb}-1$	$\Gamma_{abbb}-\rho_{ab}$	$\Gamma_{bbbb}-1$
w_a^3	Γ_{aaaa}	Γ_{aaab}			
$w_a^2 w_b$	Γ_{aaab}	Γ_{aabb}			
$w_a w_b^2$	Γ_{aabb}	Γ_{abbb}			
w_b^3	Γ_{abbb}	Γ_{bbbb}			

Note. With appropriate subscripts var $w = \sigma^2$, cov $(w_a, w_b) = \rho_{ab}\sigma^2$, ave $w^3 = \gamma\sigma^3$, ave $w^4 = \Gamma\sigma^4$.

Table 13

Coefficients of $\sigma_a^{i+m+p}\sigma_b^{j+n+q}$ in

$\text{cok}\ (w_a^i w_b^j,\ w_a^m w_b^n,\ w_a^p w_b^q)$

when w_a and w_b need not be statistically independent and each averages zero

1	2	3 w_a	w_b	w_a^2	$w_a w_b$	w_b^2
w_a	w_a	γ_{aaa}	γ_{aab}	$\Gamma_{aaaa}-1$	$\Gamma_{aaab}-\rho_{ab}$	$\Gamma_{aabb}-1$
w_a	w_b	γ_{aab}	γ_{abb}	$\Gamma_{aaab}-\rho_{ab}$	$\Gamma_{aabb}-\rho_{ab}^2$	$\Gamma_{abbb}-\rho_{ab}$
w_b	w_b	γ_{abb}	γ_{bbb}	$\Gamma_{aabb}-1$	$\Gamma_{abbb}-\rho_{ab}$	$\Gamma_{bbbb}-1$

Note. With appropriate subscripts var $w = \sigma^2$, cov $(w_a, w_b) = \rho_{ab}\sigma^2$, ave $w^3 = \gamma\sigma^3$, ave $w^4 = \Gamma\sigma^4$.

$$
\begin{aligned}
\text{var } z = \; & \Sigma h_a^{2}\sigma_a^{2} + 2\,\Sigma^* h_a h_b \rho_{ab}\sigma_a\sigma_b \\
& + \Sigma h_a h_{aa}\gamma_{aaa}\sigma_a^{3} + \Sigma h_a h_{bb}\gamma_{abb}\sigma_a\sigma_b^{2} \\
& \quad + 2\Sigma^* h_a h_{ab}\gamma_{aab}\sigma_a^{2}\sigma_b + 2\Sigma^* h_a h_{bc}\gamma_{abc}\sigma_a\sigma_b\sigma_c \\
& + \tfrac{1}{4}\,\Sigma h_{aa}^{2}\,(\Gamma_{aaaa}-1)\sigma_a^{4} + \tfrac{1}{2}\,\Sigma^* h_{aa}h_{bb}\,(\Gamma_{aabb}-1)\sigma_a^{2}\sigma_b^{2} \\
& \quad + \Sigma^* h_{aa}h_{ab}(\Gamma_{aaab}-\rho_{ab})\sigma_a^{3}\sigma_b + \Sigma^* h_{aa}h_{bc}(\Gamma_{aabc}-\rho_{bc})\sigma_a^{2}\sigma_b\sigma_c \\
& \quad + \Sigma^* h_{ab}^{2}(\Gamma_{aabb}-\rho_{ab}^{2})\sigma_a^{2}\sigma_b^{2} + 2\Sigma^* h_{ab}h_{ac}(\Gamma_{aabc}-\rho_{ab}\rho_{ac})\sigma_a^{2}\sigma_b\sigma_c \\
& \quad + 2\Sigma^* h_{ab}h_{cd}\,(\Gamma_{abcd}-\rho_{ab}\rho_{cd})\,\sigma_a\sigma_b\sigma_c\sigma_d \\
& + \tfrac{1}{3}\,\Sigma h_a h_{aaa}\Gamma_{aaaa}\sigma_a^{4} + \tfrac{1}{3}\,\Sigma^* h_a h_{bbb}\Gamma_{abbb}\sigma_a\sigma_b^{3} \\
& \quad + \Sigma^* h_a h_{aab}\Gamma_{aaab}\sigma_a^{3}\sigma_b + \Sigma^* h_a h_{abb}\Gamma_{aabb}\sigma_a^{2}\sigma_b^{2} \\
& \quad + \Sigma^* h_a h_{bbc}\Gamma_{abbc}\sigma_a\sigma_b^{2}\sigma_c + 2\Sigma^* h_a h_{abc}\Gamma_{aabc}\sigma_a^{2}\sigma_b\sigma_c \\
& \quad + 2\Sigma h_a h_{bcd}\,\Gamma_{abcd}\sigma_a\sigma_b\sigma_c\sigma_d \\
& + \text{terms of order} \geq \sigma^{5}
\end{aligned}
$$

$$
\begin{aligned}
\text{ske } z = \; & \Sigma h_a^{\,3}\gamma_{aaa}\sigma_a^{\,3} \\
& + 3\Sigma^* h_a^{\,2} h_b \gamma_{aab}\sigma_a^{\,2}\sigma_b \\
& + 6\Sigma^* h_a h_b h_c \gamma_{abc}\sigma_a\sigma_b\sigma_c \\
& + \tfrac{3}{2}\,\Sigma h_a^{\,2} h_{aa}\,(\Gamma_{aaaa}-1)\sigma_a^{\,4} + \tfrac{3}{2}\,\Sigma h_a^{\,2} h_{bb}(\Gamma_{aabb}-1)\sigma_a^{\,2}\sigma_b^{\,2} \\
& + 3\Sigma^* h_a h_b h_{aa}(\Gamma_{aaab}-\rho_{ab})\sigma_a^{\,3}\sigma_b + 3\Sigma^* h_a h_b h_{cc}(\Gamma_{abcc}-\rho_{ab})\sigma_a\sigma_b\sigma_c^{\,2} \\
& + 3\Sigma^* h_a^{\,2} h_{ab}(\Gamma_{aaab}-\rho_{ab})\sigma_a^{\,3}\sigma_b + 3\Sigma^* h_a^{\,2} h_{bc}(\Gamma_{aabc}-\rho_{bc})\sigma_a^{\,2}\sigma_b\sigma_c \\
& + 6\Sigma^* h_a h_b h_{ab}(\Gamma_{aabb}-\rho_{ab}^{\,2})\sigma_a^{\,2}\sigma_b^{\,2} + 6\Sigma^* h_a h_b h_{ac}(\Gamma_{aabc}-\rho_{ab}\rho_{ac}) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma_a^{\,2}\sigma_b\sigma_c \\
& + 6\Sigma^* h_a h_b h_{cd}\,(\Gamma_{abcd}-\rho_{ab}\rho_{cd})\sigma_a\sigma_b\sigma_c\sigma_d \\
& + \text{terms of order} \geq \sigma^5
\end{aligned}
$$

$$
\begin{aligned}
\text{elc } z = \; & \Sigma h_a^{\,4}\,(\Gamma_{aaaa}-3)\sigma_a^{\,4} \\
& + 4\Sigma^* h_a^{\,3} h_b(\Gamma_{aaab}-3\rho_{ab})\sigma_a^{\,3}\sigma_b + 6\Sigma^* h_a^{\,2} h_b^{\,2}(\Gamma_{aabb}-2\rho_{ab}^{\,2}-1)\sigma_a^{\,2}\sigma_b^{\,2} \\
& + 12\Sigma^* h_a^{\,2} h_b h_c(\Gamma_{aabc}-\rho_{bc}-2\rho_{ab}\rho_{ac})\sigma_a^{\,2}\sigma_b\sigma_c \\
& + 24\Sigma^* h_a h_b h_c h_c\,(\Gamma_{abcd}-\rho_{ab}\rho_{cd}-\rho_{ac}\rho_{bd}-\rho_{ad}\rho_{bc})\sigma_a\sigma_b\sigma_c\sigma_d \\
& + \text{terms of order} \geq \sigma^5
\end{aligned}
$$

49. Supplementary glossary and notation

The only major change over earlier parts is the amplification and extension of the notation for higher reduced (= standardized) moments. We now use γ_{aaa} as well as γ_a for ave $(w_a-\overline{w}_a)^3/\sigma_a{}^3$, and Γ_{aaaa} as well as Γ_a for ave $(w_a-\overline{w}_a)^4/\sigma_a{}^4$. We extend the fuller notation to

$$\gamma_{abc} = \text{ave} \quad (w_a-\overline{w}_a)(w_b-\overline{w}_b)(w_c-\overline{w}_c)/\sigma_a\sigma_b\sigma_c$$

and

$$\Gamma_{abcd} = \text{ave} \quad (w_a-\overline{w}_a)(w_b-\overline{w}_b)(w_c-\overline{w}_c)(w_d-\overline{w}_d) \quad /\sigma_a\sigma_b\sigma_c\sigma_d$$

and their specializations.